Name:…………………………………………………………………………………….

Student Number: ………………………………………………………………………..

Test 6 on WPPH16001.2018-2019
"Electricity and Magnetism"

Content: 12 pages (including this cover page)

Wednesday June 19 2019; A. Jacobshal 01, 9:00-12:00

- Write your full name and student number in the place above
- Write your answers in the designated areas
- Read the questions carefully
- Compose your answers is such a way that it is well indicated which (sub)question they address
- Reversed sides of each page are left blank intentionally and could be used for draft answers
- Do not use a red pen (it's used for grading) or a pencil
- Books, notes, phones, tablets, smartwatches and headphones are not allowed. Calculators and dictionaries are allowed.

Exam drafted by (name first examiner) Maxim S. Pchenitchnikov
Exam reviewed by (name second examiner) Steven Hoekstra

For administrative purposes; do NOT fill the table

The weighting of the questions:

	Maximum points	Points scored
Question 1	10	
Question 2	10	
Question 3	10	
Question 4	10	
Question 5	10	
Total	**50**	

Grade = 1 + 9 x (score/max score).

Grade: __________

Question 1. (10 points)
The index of refraction of diamond is $n_2 = 2.42$. For the air/diamond interface, calcu1ate (assume $\mu_1 = \mu_2 = \mu_0$):

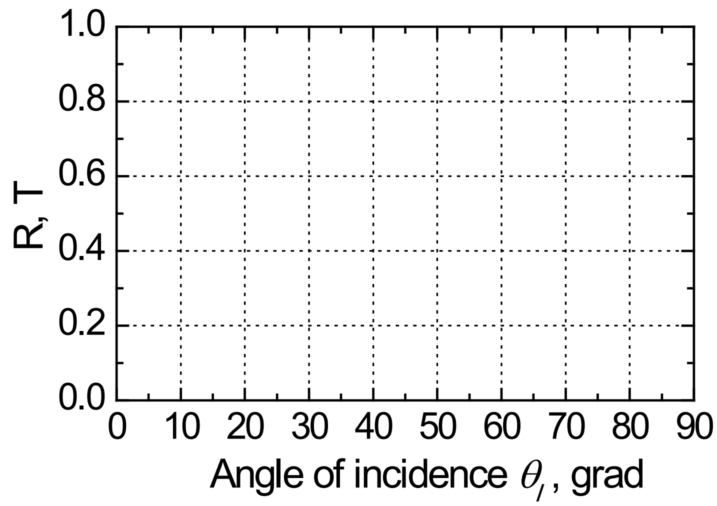
1. Reflection and transmission coefficients at normal incidence (provide the numbers!) (3 points)

2. Brewster's angle for the polarization in the plane of incidence (provide the number!) (1 point)

3. What is the value of the reflection coefficient at 90° (provide the number!) (1 point)

4. In the panel below draw schematically (but realistically, using the values calculated) dependences of reflection and transmission coefficients on the incident angle for the polarization in the plane of incidence. (5 points)

Answers:



Answers to Question 1 (Problem 9.18 modified) (10 points)

From the extended formula sheet:

$$R = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2 ; \; T = \alpha\beta\left(\frac{2}{\alpha + \beta}\right)^2 \quad \left[\alpha \equiv \frac{cos\theta_T}{cos\theta_I} = \frac{\sqrt{1 - (\sin\theta_I / n_2)^2}}{cos\theta_I}; \; \beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = n_2\right]$$

1. 3 points (-1 point if the numerical value is not correct)

$\theta_I = 0; \; \alpha = 1$

$$R = \left(\frac{1 - n_2}{1 + n_2}\right)^2 = \left(\frac{1 - 2.42}{1 + 2.42}\right)^2 = 0.17; T = 1 - R = 0.83$$

2. 1 point

$\theta_B = atan(n_2) = 67.5°$

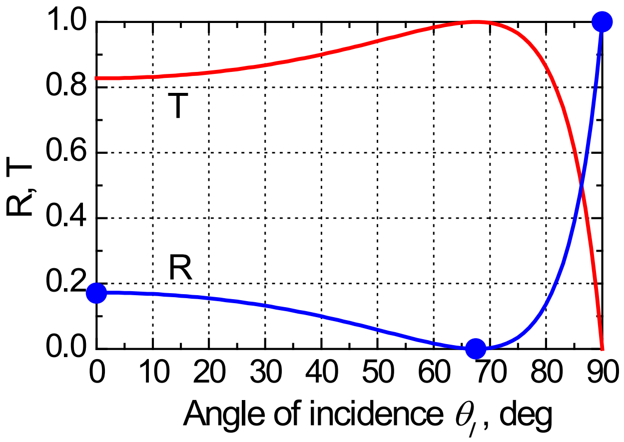
3. $R = 1$ **(1 point)**

4. 5 points

-1 point if $R + T \neq 1$

3 points if only one of the curves is depicted

-1/2 point for each reference value wrongly shown



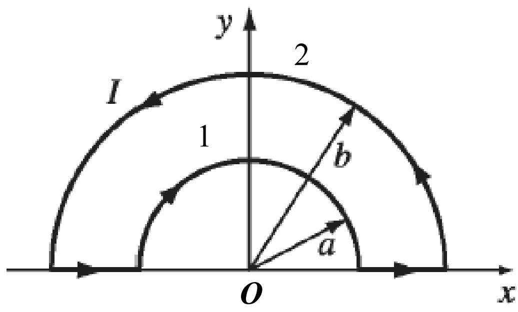
Notes:

1. At Brewster's angle the reflection of this polarization in zero. Therefore, the reflection coefficient must have a minimum (zero) at 67.5°.
2. "atan" is not "*a* tangent"; it stands for the inverse tangent, aka "arctangent".

Question 2. (10 points)

A piece of wire bent into a loop, as shown in the figure, carries a current that increases linearly with time:

$I = kt \ \ (-\infty < t < \infty)$



1. Show that the retarded vector potential $\vec{\mathbf{A}}$ at the center *O* is

$\vec{\mathbf{A}} = \dfrac{\mu_0 kt}{2\pi} ln(b/a)\, \hat{\mathbf{x}}$ (7 points)

2. Find the electric field $\vec{\mathbf{E}}$ at the center. (1 point)

3. Explain why this electrically-neutral wire produces an *electric* field. (1 point)

4. Can you determine the *magnetic* field from the expression for $\vec{\mathbf{A}}$ derived in exercise (1)? Explain your answer. (1 point)

Answers:

Answers to Question 2 (Problem 10.12) (10 points)
1. 7 points

$$\vec{\mathbf{A}} = \frac{\mu_0}{4\pi}\int \frac{\vec{\mathbf{I}}(t_r)}{\mathscr{r}} = \frac{\mu_0}{4\pi}\int \frac{(t-\mathscr{r}/c)}{\mathscr{r}} d\vec{\mathbf{l}} = \frac{\mu_0}{4\pi}\left\{t\int \frac{d\vec{\mathbf{l}}}{\mathscr{r}} - \frac{1}{c}\int d\vec{\mathbf{l}}\right\}$$ 2 points

For the complete loop, $\oint d\vec{\mathbf{l}} = 0$ 1 point

$$\vec{\mathbf{A}} = \frac{\mu_0 kt}{4\pi}\left\{\int_1 \frac{d\vec{\mathbf{l}}}{a} + \int_2 \frac{d\vec{\mathbf{l}}}{b} + 2\,\hat{\mathbf{x}}\int_a^b \frac{dx}{x}\right\}$$

Inner semicircle: $\int_1 d\vec{\mathbf{l}} = 2a\,\hat{\mathbf{x}}$ (the integral in the y-direction goes up and down, i.e. = 0) 1 point

Outer semicircle: $\int_2 d\vec{\mathbf{l}} = -2b\,\hat{\mathbf{x}}$ (mind the sign!) 1 point

$$\vec{\mathbf{A}} = \frac{\mu_0 kt}{4\pi}\left\{\frac{2a}{a} - \frac{2b}{b} + 2ln(b/a)\right\}\hat{\mathbf{x}};\ \vec{\mathbf{A}} = \frac{\mu_0 kt}{2\pi} ln(b/a)\,\hat{\mathbf{x}}$$ 2 points (- 1 point if no vectors)

2. 1 point

$V = 0$

$$\vec{\mathbf{E}} = -\frac{\partial\vec{\mathbf{A}}}{\partial t} = \frac{\mu_0 k}{2\pi} ln(b/a)\,\hat{\mathbf{x}}$$ (- 1/2 point if no vectors)

3. 1 point

The changing magnetic field induces the electric field

4. Since we only know $\vec{\mathbf{A}}$ at one point (the center), we can't compute $\vec{\mathbf{B}} = \nabla \times \vec{\mathbf{A}}$ to get $\vec{\mathbf{B}}$. 1 point

Notes added:

1. Many forgot that the time in the first line is the **retarded** time (even though the integral containing the retarded time was correctly copied from the formula sheet). Curiously enough, this does not nesseseraly lead to the wrong answer as the retarded-time integral is zero (the second line above). Nonetheless, this is an error for which points were deducted.
2. Some of those who wrote the retarded time correctly, assumed that $d\vec{\mathbf{l}}$ is a **scalar** (it's a vector of course). However, the close-loop integral over the scaler should not be zero. Here the second error was made by assuming that it is zero so that one error compensated another. Which keeps them errors, of course.
3. Some simply mixed up the volume integral in the retarded potential formula and the fact that here the currents are one dimentional. Of course, by adding a factor of two here and there you can arrive at the provided answer but this does not make your solution correct.

Question 3. (10 points)

A dipole oscillates with frequency ω along the z-direction.

1. Using Maxwell's equations, find the magnetic field $\vec{\mathbf{B}}$ from the known electric field $\vec{\mathbf{E}}$ in the radiation zone. (6 points)

2. Calculate the Poynting vector $\vec{\mathbf{S}}$. (1 point)

3. Calculate the time-averaged intensity $\vec{\mathbf{I}} = \langle \vec{\mathbf{S}} \rangle$, emitted by the dipole. (1 point)

4. Prove that the total power radiated does not depend on the radius of the sphere over which the power is calculated (you don't have to calculate the power). (2 points)

Answers to Question 3 (Problem 9.35 simplified, and lectures) (10 points)

1. (6 points)

From the formula sheet,

$$\vec{\mathbf{E}} = A\left(\frac{sin\theta}{r}\right)cos[\omega(t - r/c)]\,\widehat{\boldsymbol{\theta}}, \text{with } A = -\frac{\mu_0 p_0 \omega^2}{4\pi}$$ 1 point

From Faraday's equation: $\frac{\partial \vec{\mathbf{B}}}{\partial t} = -\nabla \times \vec{\mathbf{E}}$

$\vec{\mathbf{E}}$ has only the $\widehat{\boldsymbol{\theta}}$ component. In spherial coordinates,

$$-\nabla \times \vec{\mathbf{E}} = -\frac{1}{r}\frac{\partial}{\partial r}(rE_\theta)\,\widehat{\boldsymbol{\phi}} = -\frac{1}{r}\frac{\partial}{\partial r}\left(rA\left(\frac{sin\theta}{r}\right)cos[\omega(t - r/c)]\right)\widehat{\boldsymbol{\phi}}$$ 2 points

$$= -\frac{Asin\theta}{r}(-sin[\omega(t - r/c)])\left(-\frac{\omega}{c}\right)\widehat{\boldsymbol{\phi}} = -\frac{A\omega}{c}\left(\frac{sin\theta}{r}\right)sin[\omega(t - r/c)]\,\widehat{\boldsymbol{\phi}}$$

$$\vec{\mathbf{B}} = -\frac{A\omega}{c}\left(\frac{sin\theta}{r}\right)\int sin[\omega(t - r/c)]\,dt\;\widehat{\boldsymbol{\phi}} = -\frac{A\omega}{c}\left(\frac{sin\theta}{r}\right)(-cos[\omega(t - r/c)])\frac{1}{\omega}\,\widehat{\boldsymbol{\phi}}$$ 2 points

$$\vec{\mathbf{B}} = \frac{A}{c}\left(\frac{sin\theta}{r}\right)\cos[\omega(t - r/c)]\;\widehat{\boldsymbol{\phi}}$$ 1 point

Alternatively, $\nabla \times \vec{\mathbf{B}} = \mu_0\epsilon_0\frac{\partial \vec{\mathbf{E}}}{\partial t}$

$$-\frac{1}{r}\frac{\partial}{\partial r}(rB_\varphi) = \mu_0\epsilon_0\frac{Asin\theta}{r}(-sin[\omega(t - r/c)])\,\omega$$ 2 points

$$rB_\varphi = \mu_0\epsilon_0 A\,sin\theta(-cos[\omega(t - r/c)])\left(-\frac{c}{\omega}\right)\omega$$ 2 points

$$\vec{\mathbf{B}} = \frac{A}{c}\left(\frac{sin\theta}{r}\right)\cos[\omega(t - r/c)]\;\widehat{\boldsymbol{\phi}}$$ 1 point

2. (1 point)

$$\vec{\mathbf{S}} = \frac{1}{\mu_0}(\vec{\mathbf{E}} \times \vec{\mathbf{B}}) = \frac{\mu_0 p_0^2 \omega^4}{16\pi^2 c}\left(\frac{sin^2\theta}{r^2}\right)\cos^2[\omega(t - r/c)]\,\hat{\mathbf{r}}$$ 1 point

3. (1 point) 1 point

$$\vec{\mathbf{I}} = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\left(\frac{sin^2\theta}{r^2}\right)\hat{\mathbf{r}}$$

4. (2 points)

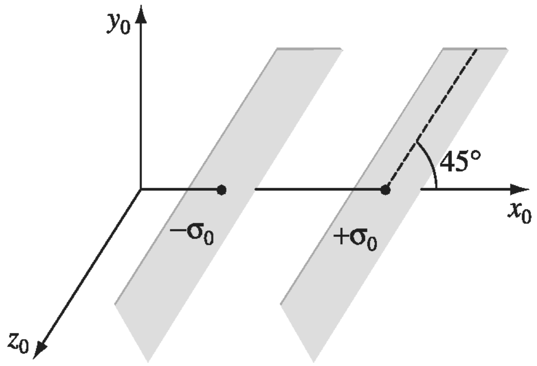
$$\langle P\rangle = \int\langle\vec{\mathbf{I}}\rangle\cdot d\mathbf{a} = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\int\frac{sin^2\theta}{r^2}r^2 sin\theta\,d\theta d\varphi = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\int sin^3\theta\,d\theta d\varphi$$ 2 points

i.e. the r-dependence cancels

Notes added:

When calculated $\langle P\rangle$, many take $\langle\vec{\mathbf{I}}\rangle$ out of the integral thereby neglecting the r^{-2} dependence. Simultaneously, they forget about the r^2 dependence in $d\mathbf{a}$ so that one mistake compensates for another, and $\langle P\rangle$ has no r-dependence. Which doesn't make mistakes no-mistakes, of course, and doesn't make the answer correct.

Question 4. (10 points)



A parallel-plate capacitor, at rest in $\mathcal{S}_0$ and tilted at a 45° angle to the x_0 axis, carries charge densities $\pm\sigma_0$ on the two plates (see figure). System $\mathcal{S}$ is moving to the right at speed v relative to $\mathcal{S}_0$.

1. Show that the electric field $\vec{\mathbf{E}}_0$ in $\mathcal{S}_0$ is

$$\vec{\mathbf{E}}_0 = \frac{\sigma_0}{\sqrt{2}\epsilon_0}(-\,\hat{\mathbf{x}} + \hat{\mathbf{y}}) \qquad \text{(1 point)}$$

2. Show that the electric field $\vec{\mathbf{E}}$ in $\mathcal{S}$ is

$$\vec{\mathbf{E}} = \frac{\sigma_0}{\sqrt{2}\epsilon_0}(-\,\hat{\mathbf{x}} + \gamma\hat{\mathbf{y}}) \qquad \text{(2 points)}$$

3. Show that the angle the plates make with the x axis in system $\mathcal{S}$ is $\theta = \tan^{-1}\gamma$. (1 point)

4. Show that cosine of the angle ϕ between the field and a unit vector $\hat{\mathbf{n}}$ perpendicular to the plates, in $\mathcal{S}$, is

$$\cos\phi = \frac{2\gamma}{1+\gamma^2}$$

i.e. the electric field is ***not*** perpendicular to the plates in $\mathcal{S}$. (6 points)

Tip 1. The unit vector perpendicular to the plates in $\mathcal{S}$, is expressed as $\hat{\mathbf{n}} = -\sin\theta\ \hat{\mathbf{x}} + \cos\theta\ \hat{\mathbf{y}}$

Tip 2. Calculate $\cos\phi$ via the scalar product $\vec{\mathbf{E}}\cdot\hat{\mathbf{n}}$ as $\cos\phi = \left(\vec{\mathbf{E}}\cdot\hat{\mathbf{n}}\right)/|E|$

Answers to Question 4 (Problem 12.43) (10 points)

1. (1 point)

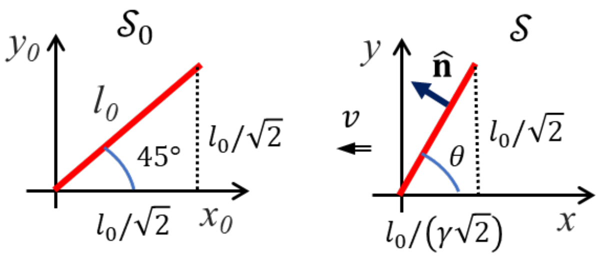
$$\vec{\mathbf{E}}_0 = \frac{\sigma_0}{\epsilon_0}(-cos45°\,\hat{\mathbf{x}} + cos45°\,\hat{\mathbf{y}}) = \frac{\sigma_0}{\sqrt{2}\epsilon_0}(-\hat{\mathbf{x}} + \hat{\mathbf{y}})$$ 1 points (-1/2 if the signs are not correct

2. (2 points)

$$E_x = E_{x_0} = -\frac{\sigma_0}{\sqrt{2}\epsilon_0};\ E_y = \gamma E_{y_0} = \gamma\frac{\sigma_0}{\sqrt{2}\epsilon_0}; \vec{\mathbf{E}} = \frac{\sigma_0}{\sqrt{2}\epsilon_0}(-\hat{\mathbf{x}} + \gamma\hat{\mathbf{y}})$$ 2 points (-1 if vectors are not correct)

3. (1 point)

To observer in $\mathcal{S}$, the y-dimension is unaffected, but the x-dimension is Lorentz contracted



$$\tan\theta = \frac{l_0/\sqrt{2}}{l_0/(\gamma\sqrt{2})} = \gamma;\ \theta = \tan^{-1}\gamma$$ 1 point

4. (6 points)

Let $\hat{\mathbf{n}}$ be a unit vector perpendicular to the plates in $\mathcal{S}$:

$\hat{\mathbf{n}} = -\sin\theta\ \hat{\mathbf{x}} + \cos\theta\ \hat{\mathbf{y}}$

$$|E| = \frac{\sigma_0}{\sqrt{2}\epsilon_0}\sqrt{1+\gamma^2}$$ 1 point

The angle ϕ between $\hat{\mathbf{n}}$ and $\vec{\mathbf{E}}$ is:

$$\cos\phi = \frac{\vec{\mathbf{E}}\cdot\hat{\mathbf{n}}}{|E|} = \frac{\frac{\sigma_0}{\sqrt{2}\epsilon_0}(-\hat{\mathbf{x}}+\gamma\hat{\mathbf{y}})\cdot(-\sin\theta\ \hat{\mathbf{x}} + \cos\theta\ \hat{\mathbf{y}})}{\frac{\sigma_0}{\sqrt{2}\epsilon_0}\sqrt{1+\gamma^2}} = (\sin\theta + \gamma\cos\theta)\frac{1}{\sqrt{1+\gamma^2}}$$ 2 points

$$\gamma = \tan\theta = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}};\ \gamma^2(1-\sin^2\theta) = \sin^2\theta\,; \sin\theta = \frac{\gamma}{\sqrt{1+\gamma^2}}$$ 1 point

$$\gamma = \tan\theta = \frac{\sqrt{1-\cos^2\theta}}{\cos\theta};\ \gamma^2\cos^2\theta = 1-\cos^2\theta\,; \cos\theta = \frac{1}{\sqrt{1+\gamma^2}}$$ 1 point

$$\cos\phi = \left(\frac{\gamma}{\sqrt{1+\gamma^2}} + \gamma\frac{1}{\sqrt{1+\gamma^2}}\right)\frac{1}{\sqrt{1+\gamma^2}} = \frac{2\gamma}{1+\gamma^2}$$ 1 point

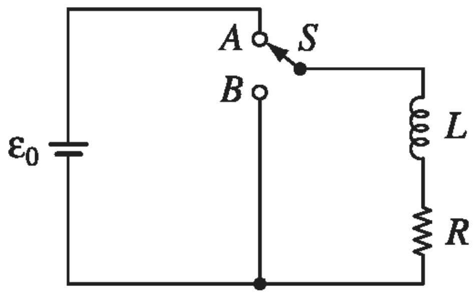
Evidently, the field is ***not*** perpendicular to the plates in $\mathcal{S}$.

Notes added:

1. The dot product is a scalar not a vector so that $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ should not be present
2. In calculating $|E|$, many forget the square root in $\sqrt{1+\gamma^2}$ (writing instead $1+\gamma^2$)
3. Saying "$\gamma = \tan\theta$ so that $\theta = \tan^{-1}\gamma$" does not prove anything!

Question 5. (10 points)

Suppose the circuit in the figure (consisting of a battery with emf of $\mathcal{E}_0$, resistor R and inductivity coil L) has been connected for a long time when suddenly, at time $t = 0$, switch S is thrown from A to B, bypassing the battery.



1. Considering the back emf of the inductivity coil as an emf source in the S-B-R-L circuit, show that the current at any subsequent time t is

$I(t) = \frac{\mathcal{E}_0}{R} e^{-Rt/L}$ (4 points)

2. What is the power delivered to the resistor, as a function of time? (1 point)
3. By integration the power over time, find the total energy delivered to the resistor. (3 points)
4. Where was this energy stored? Prove your answer. (2 points)

Answers to Question 5 (Problem 7.31) (10 points)

1. 4 points

$$-L\frac{dI}{dt} = IR;\ \frac{dI}{I} = -\frac{R}{L}dt; \ln I - \ln I_o = -\frac{R}{L}t;\ I(t) = I_o e^{-\frac{Rt}{L}}$$ 3 points

At $t = 0, I = \frac{\mathcal{E}_0}{R}$ so that $I_o = \frac{\mathcal{E}_0}{R}$ 1 point

$$I(t) = \frac{\mathcal{E}_0}{R} e^{-Rt/L}$$

2. Power delivered

$$P = I^2 R = \frac{\mathcal{E}_0^2}{R} e^{-2Rt/L}$$ 1 point

3. Energy delivered

$$\frac{dW}{dt} = P; W = \int_0^\infty P\, dt = \frac{\mathcal{E}_0^2}{R}\int_0^\infty e^{-2Rt/L}\, dt = \frac{\mathcal{E}_0^2}{R}\left(-\frac{L}{2R}e^{-2Rt/L}\right)\Bigg|_0^\infty = \frac{L}{2}\frac{\mathcal{E}_0^2}{R^2}$$ 3 points

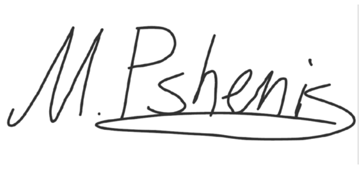
4. 2 points

The energy was stored in the induction coil (1 point) as

$$W_0 = \frac{1}{2}LI_0^2 = \frac{L}{2}\frac{\mathcal{E}_0^2}{R^2}$$ 1 point

Notes added:

1. While calculating the energy, the integral should run from t=0 to infinity.
2. The energy was indeed initially stored in the battery (and before that in gas/oil, and before that in the Sun etc) but the question is about what happens from the zero time on, i.e. when the switch was thrown from A to B.
3. Many answered that the energy **is** stored in the resistor (which is true: the energy is stored in motions of the lattice of the material the resistor is made of). However, the question was where the energy **was** stored before it ended up at the resistor (and after the switch was flipped). Both answers “magnetic field” and “induction coil” are correct provided that they are supported by the respective equations.
4. Many simply did not bother to prove their statement where the energy was stored. It’s a pity because this cost one point.



Maxim Pchenitchnikov

June 12 2019

Steven Hoekstra